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Filtering Bases and Cohomology of Nilpotent Subalgebras of the Witt Algebra and the Algebra of Loops in sl_2

F. V. Weinstein

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ABSTRACT. We study the cohomology with trivial coefficients of the Lie algebras L_k, $k \geqslant 1$, of polynomial vector fields with zero k-jet on the circle and the cohomology of similar subalgebras $\mathscr{L}_k$ of the algebra of polynomial loops with values in sl_2. The main result is a construction of special bases in the exterior complexes of these algebras. Using this construction, we obtain the following results. We calculate the cohomology of L_k and $\mathscr{L}_k$. We obtain formulas in terms of Schur polynomials for cycles representing the homology of these algebras. We introduce "stable" filtrations of the exterior complexes of L_k and $\mathscr{L}_k$, thus generalizing Goncharova's notion of stable cycles for L_k, and give a polynomial description of these filtrations. We find the spectral resolutions of the Laplace operators for L_1 and $\mathscr{L}_1$.

Dedicated to the hallowed memory of my mother, Maria Weinstein

Let $h^3 = 1$, where $h \in \mathbb{C}$. Denote by $L^{(h)}$ the Lie algebra spanned over the field $\mathbb{Q}$ of rationals by finite linear combinations of basis vectors e_a, where a runs over the ring $\mathbb{Z}$ of integers and the commutator is defined by the formula

$$[e_a, e_b] = (b-a)_h\, e_{a+b}, \quad \text{where } (m)_h = \frac{h^{2m} - h^m}{h^2 - h},\ m \in \mathbb{Z}. \tag{1}$$

The algebra $L^{(1)}$ is isomorphic to the algebra spanned by the derivations $e_a = t^{a+1} d/dt$ of the Laurent polynomial ring $\mathbb{Q}[t^{-1}, t]$. This Lie algebra is called the *Witt algebra*. The Lie algebra $L^{(h\neq 1)}$ is isomorphic to the loop algebra $sl_2(\mathbb{Q}) \otimes \mathbb{Q}[t^{-1}, t]$ on $sl_2(\mathbb{Q})$ (see [8, Exercise 7.12]).

It seems to me that one cause for numerous analogies between these two algebras is the possibility to represent their commutators in the parametric form (1). For example, each of these algebras has a unique (up to an isomorphism) universal central extension, and the 2-cocycles determining these central extensions can be represented in the unified form

$$(e_i, e_j) \to \delta_{i+j,0}\, \kappa_h(i), \quad \text{where } \kappa_h(i) = \frac{1}{2}\sum_{r=1}^{i} (i - 2r + 1)_h^2 = \begin{cases} \binom{i+1}{3} & \text{if } h = 1, \\ \lfloor \frac{i+1}{3} \rfloor & \text{if } h \neq 1. \end{cases} \tag{2}$$

(Here $\lfloor a \rfloor$ is the integer part of a number a.) For $h = 1$, this is the Gelfand–Fuchs cocycle, and the corresponding extension is called the Virasoro algebra. For $h \neq 1$, we obtain the affine algebra $A_1^{(1)}$.

Let $L^{(h)}(k) \subset L^{(h)}$ be the subalgebra generated by the vectors e_a with $a \geqslant k \geqslant -1$. We sometimes write $L_k := L^{(1)}(k)$ and $\mathscr{L}_k := L^{(h\neq 1)}(k)$. For $k \geqslant 1$, these algebras are nilpotent. The main result of this paper is a construction of special bases in the exterior complexes of the algebras $L^{(h)}(k)$ for $k \geqslant 1$. One application of this construction is the computation of the continuous cohomology spaces $H^*(L^{(h)}(k); \mathbb{Q})$, where $\mathbb{Q}$ is treated as a trivial $L^{(h)}(k)$-module and continuous $\mathbb{Q}$-linear functionals on $L^{(h)}(k)$ are those with finite-dimensional supports. In particular, we prove that

$$\sum_{q=0}^{\infty} \dim H^q(L^{(h)}(k); \mathbb{Q})\, t^q = \frac{1+t}{(1-t)^k}\,. \tag{3}$$

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For $h=1$, this formula (in a different form; see Remark 2.6) was included in Gelfand's talk [5] as a conjecture.* The first published proof of the conjecture by Goncharova [7] is very cumbersome. Later, several proofs of her theorem on the basis of different, very interesting ideas were found for $k=1$ (e.g., see [2] and [11]).**

Our work is motivated by two sources. The first one is Goncharova's paper [7], where she introduced the definitions, important to us, of *non-singular* and *main k-partitions* and *stable cycles* in the exterior complexes of the algebras L_k. Goncharova's main result is a proof of the existence of a stable cycle for each main k-partition and of the fact that the homological classes of these cycles form a basis of the homology space $H_*(L_k;\mathbb{C})$.

The second source is the paper [6] by Gelfand, Feigin, and Fuchs. On the exterior complex of the algebra L_1, they consider the Euclidean structure in which the vectors $e_{i_1}\wedge\cdots\wedge e_{i_q}$ are orthonormal. Then there is a linear self-adjoint "Laplace operator" Γ naturally defined on the complex. The kernel of Γ is isomorphic to the homology of L_1. The main, quite unexpected observation in [6] is that the eigenvalues of this operator are explicitly computable integers. Additionally, a nice description of the stable cycles of L_k is given.

Unfortunately, the paper [6] is not completely correct. Namely, the main tool of analysis in [6]—the construction proposed there for the eigenvectors of Γ—often fails. As a consequence, the statement of the main result contains an error: the claimed multiplicities of the eigenvalues of Γ are wrong, although the eigenvalues themselves are correct.

The desire to clarify the ideas and results in [6] and [7] and understand the relations between them has been the main motivation for me. The paper is organized as follows.

Section 1 gathers some material pertaining to partitions and used throughout the paper. The main result is stated in Section 2. Namely, in terms of the marked k-partitions introduced in Section 1, we define $\varepsilon(k)$-monomials as certain vectors in the exterior complex $C_*(k)$ of $L^{(h)}(k)$. The main Theorem 2.4 claims that these vectors form a basis of the complex and that this basis has some "filtering" properties. Then, taking this theorem for granted, we compute the cohomology space $H^*(L^{(h)}(k);\mathbb{Q})$.

The proof of the main theorem is contained in Section 3. First, we establish some identities, playing a key role in what follows, in the exterior complexes of $L^{(h)}(k)$ (Lemma 3.1). We use these identities to prove that the $\varepsilon(1)$-monomials generate $C_*(1)$ and have the properties claimed in Theorem 2.4. The linear independence of the $\varepsilon(1)$-monomials follows from a formula that expresses the number of integer partitions of positive integers with pairwise distinct parts via such monomials. This formula proves to be equivalent to Sylvester's classical identity

$$\prod_{i=1}^{\infty}(1+tx^i) = 1 + tx\,\frac{1+tx^2}{1-x} + \sum_{l=2}^{\infty} t^l x^{(3l^2-l)/2}\,\frac{(1+tx)\cdots(1+tx^{l-1})(1+tx^{2l})}{(1-x)\cdots(1-x^{l-1})(1-x^l)} \tag{4}$$

in partition theory (see [12, p. 282]). The proof of the theorem is completed by induction on k with the use of the above-mentioned identities and an additional construction for marked partitions.

It seems to be quite interesting that Sylvester's identity arises when constructing the filtering basis. But it is somewhat disappointing that we rely on this identity in the proof of the main theorem. Therefore, in the additional Section 8 I give a bijective proof of the desired combinatorial fact that is shown in Section 3 to be equivalent to this identity. This equivalence provides a new interpretation of Sylvester's identity.

The subsequent sections contain applications of the main theorem. In Section 4, we introduce two filtrations of the complex $C_*(k)$, an increasing "stable" filtration whose initial term is the space of stable cycles and a decreasing ε-filtration. The definition of the stable filtration uses only the definition of the complex $C_*(k)$, while to define the ε-filtration we need the main theorem. In Section 4, we show that these filtrations are dual with respect to the Euclidian structure considered.

* As far as I know, this conjecture is due to D. B. Fuchs.

** In [2], the cases of $k=2,3$ are considered as well, but the treatment is *ad hoc* and cannot be generalized to $k>3$.

In Section 5, we follow an idea in [6] and identify the chains of algebras $L^{(h)}(k)$ with skew-symmetric polynomials. Using this identification, we obtain a formula for the action of the boundary operator in the exterior complex. This formula generalizes the corresponding formula in [6] to $L^{(h)}(k)$. The approach to stable cycles suggested in [6] is generalized as well and gives a polynomial description of the stable filtration.

A comparison of the results in Sections 4 and 5 leads to a family of bases in the symmetric polynomial rings with finitely many variables (Corollary 5.4). Such a basis depends on the parameters h and k, consists of homogeneous polynomials, and is numbered by marked k-partitions. For $k = 1$, this gives a combinatorial formula for the dimension of the space of homogeneous polynomials $f(t_1, \dots, t_q)$ of fixed degree over $\mathbb{Q}$ with the property

$$f(ht_1, \overline{h}t_1, \dots, ht_m, \overline{h}t_m, t_{m+1}, \dots, t_{q-m}) = 0,$$

where $0 \leqslant m \leqslant \lfloor q/2 \rfloor$, $h^3 = 1$, and $\overline{h}$ is the complex conjugate of h. This dimension is independent of the choice of h.

In Section 6, the results obtained are used for computing the homology of $L^{(h)}(k)$. First, we build a basis of $H_*(L^{(h)}(k); \mathbb{Q})$ represented by stable cycles, which extends Goncharova's result to the algebras $L^{(h)}(k)$. Next, we obtain explicit formulas for these cycles via Schur polynomials.

In Section 7, we consider the Laplace operator $\Gamma^{(h)}$ for the algebra $L^{(h)}(1)$. We prove that the action of $\Gamma^{(h)}$ is expressed in the basis of $\varepsilon(1)$-monomials by a rational triangular matrix whose diagonal entries can be computed explicitly. In particular, this gives the spectrum of $\Gamma^{(h)}$. Using this, we show that the multiplicities of the eigenvalues of $\Gamma^{(1)}$ are finite and their computation can be reduced to a number-theoretic problem. In the basis of $\varepsilon(1)$-monomials, the action of $\Gamma^{(h\neq 1)}$ is diagonal, the spectrum of $\Gamma^{(h\neq 1)}$ is the set $\mathbb{Z}_{\geqslant 0}$, and the multiplicity of each eigenvalue is infinite. One corollary of our calculations is the invariance of the stable filtration with respect to the action of $\Gamma^{(h)}$. This generalizes the claim in [6] concerning the action of the Laplacian on the space of stable cycles of the algebra L_1. For $h \neq 1$, our result on the spectral resolution of $\Gamma^{(h)}$ is a well known special case of a much more general fact (see [9, 3.2–3.4]).

Many interesting questions about the cohomology of the algebras $L^{(h)}(k)$ can be reduced to questions about the expansions of their cochains in the filtering basis. The problem of computing the multiplicative structure of the rings $H^*(L^{(h)}(k); \mathbb{Q})$ (see Remark 2.7) can serve as an example. The problem of describing the structure of $H^*(L^{(h)}(k); \mathbb{Q})$ as an $L^{(h)}(1)/L^{(h)}(k)$-module for $k \geqslant 2$ (see [1] and [3]) is another example. In full generality, these problems are open yet.

Part of the results presented here were already published in [13] and [14],* where the algebras L_k of vector fields were considered. In this paper, in addition to technical simplifications and new results, we simultaneously consider the algebras $L^{(h)}(k)$ for $h = 1$ and $h \neq 1$ in the framework of a unified approach and emphasize the analogy between them.

Notation. The symbols α, β, and γ (with or without subscripts) denote rational numbers. Unless otherwise specified, all vector spaces are assumed to be defined over $\mathbb{Q}$; k stands for a positive integer. The cardinality of a set M is denoted by $|M|$. For $x, y \in M$, the function $\delta_{x,y}$ is the Kronecker delta. The exterior algebra of a vector space V is denoted by $\Lambda(V) = \bigoplus_{q=0}^{\infty} \Lambda^q(V)$, where $\Lambda^q(V)$ is the qth exterior power of V. To refer to the algebras $L^{(h)}(k)$ with different h simultaneously, we write $L(k)$. For various objects related to these algebras, we usually omit h and specify which h is meant where necessary.

1. Nonsingular, Main, and Marked Partitions

This section introduces the main definitions and notation concerning partitions and marked partitions, to be used throughout the paper.

*The paper [13] contains a mistake, corrected in [14], in the definition of the spectral sequence.

Definition 1.1. A *partition* is a finite ordered set $I = [i_1, \ldots, i_q]$ of integers, referred as the *parts of the partition*, such that $0 \leqslant i_1 \leqslant \cdots \leqslant i_q$. A partition is said to be *strict* if $i_1 < \cdots < i_q$. We write $I^- = i_1$ and $I^+ = i_q$.

The numbers $\|I\| = i_1 + \cdots + i_q$ and $|I| = q$ are called the *degree* and *length* of I, respectively.

A *subpartition* of I is an ordered subset $I' \subset I$. The *union* $I_1 \sqcup I_2$ *of partitions* I_1 and I_2 is the partition whose set of parts is the disjoint union of the sets of parts of I_1 and I_2.

Definition 1.2. A *marked partition* is an equivalence class of the pairs $[I; J]$, where I is a partition, $J \subset I$, and $[I; J]$ is equivalent to $[I_1; J_1]$ if $I = I_1$ and $J = J_1$ as partitions. The elements of J are called the *marked parts*. We identify each partition I with the marked partition $[I; \varnothing]$. The numbers $\|I\|$ and $|I|$ are called the *degree* and *length* of $[I; J]$, respectively. We set $[I_1; J_1] \sqcup [I_2; J_2] = [I_1 \sqcup I_2; J_1 \sqcup J_2]$. The set of marked partitions is denoted by $\mathbf{D}$.

When this cannot lead to a confusion, we refer to marked partitions simply as partitions. Instead of explicitly indicating the set of marked parts, we often underline these parts, $[[1, 4, 6, 7]; [4, 7]] = [1, \underline{4}, 6, \underline{7}]$. Note that, by definition, one has, e.g., $[\underline{5}, 5] = [5, \underline{5}]$.

Definition 1.3. Let I and I' be distinct partitions. We write $I' \vartriangleleft I$ (I' is less than I) if $|I'| = |I|$, $\|I'\| = \|I\|$, and

$$i'_1 + \cdots + i'_r \leqslant i_1 + \cdots + i_r \quad \text{for all } r,\ 1 \leqslant r \leqslant |I|. \tag{5}$$

Definition 1.4. Let $[I; J]$ and $[I'; J']$ be distinct marked partitions. We write $[I'; J'] \vartriangleleft [I; J]$ if $|I'| + |J'| = |I| + |J|$ and either $|I'| < |I|$, or $I' \vartriangleleft I$, or $I' = I$ and $J' \prec J$, where $\prec$ stands for the lexicographic order.

For example, $[\underline{5}] \vartriangleleft [2, 3]$, $[\underline{3}, 6] \vartriangleleft [3, \underline{6}]$, and $[\underline{3}, 6, 10, \underline{12}] \vartriangleleft [3, \underline{6}, \underline{10}, 12]$.

One can readily show that $\trianglelefteqslant$ is a partial order on $\mathbf{D}$.

Claim (1.16)* in [10] can be restated as follows:

Lemma 1.5. *For integer a and b, $1 \leqslant a < b \leqslant q$, let $R_{a,b} \colon \mathbb{Z}^q \to \mathbb{Z}^q$ be the mapping defined by the formula*

$$R_{a,b}(i_1, \ldots, i_q) = (i_1, \ldots, i_a + 1, \ldots, i_b - 1, \ldots, i_q).$$

If $I' \vartriangleleft I$ and $|I'| = |I|$, then there exists a sequence of pairs (a_r, b_r) of integers, $1 \leqslant a_r < b_r \leqslant q$, such that $I' = I_1 \vartriangleleft \cdots \vartriangleleft I_m = I$, where $I_{r+1} = R_{a_r,b_r}(I_r)$.

Lemma 1.6. *If $I' \trianglelefteqslant I$ and $I'_1 \vartriangleleft I_1$, then $I' \sqcup I'_1 \vartriangleleft I \sqcup I_1$.*

Proof. It suffices to consider the case in which $I = I'$. By Lemma 1.5, it suffices to show that $I \sqcup R_{a,b}(I'_1) \vartriangleleft I \sqcup I'_1$ for a and b such that $R_{a,b}(I'_1) \vartriangleleft I'_1$. But this is obvious. □

Corollary 1.7. *If $[I'; J'] \trianglelefteqslant [I; J]$ and $[I'_1; J'_1] \vartriangleleft [I_1; J_1]$, then $[I'; J'] \sqcup [I'_1; J'_1] \vartriangleleft [I; J] \sqcup [I_1; J_1]$.*

Definition 1.8 [7]**.** A partition $[i_1, \ldots, i_q]$ is said to be *nonsingular* if $i_{r+1} - i_r \geqslant 3$, $1 \leqslant r \leqslant q - 1$. A *dense* partition is a partition of the form $\xi(a, q) = [a, a + 3, \ldots, a + 3(q - 1)]$.

Definition 1.9. A *k-partition* is a pair $\{k; I\}$, where I is a partition and $I^- \geqslant k$. A k-partition is said to be *nonsingular* if I is a nonsingular partition.

We write k-partitions as usual partitions, emphasizing that we only consider k-partitions. For example, one may treat $[2, 6]$ as either a 1-partition or a 2-partition. These objects are not the same.

Definition 1.10 [7]**.** A *main k-partition* is a nonsingular k-partition $[i_1, \ldots, i_q]$ such that

$$i_q \leqslant 2k + 3(q - 1) \quad \text{for } i_1 > k \quad \text{and} \quad i_q < 2k + 3(q - 1) \quad \text{for } i_1 = k.$$

For example, $\xi(1, q)$ and $\xi(2, q)$ exhaust all main 1-partitions of length $q \geqslant 1$.

Definition 1.11. For a nonsingular k-partition I, there exists a unique decomposition $I = I_1 \sqcup \cdots \sqcup I_s$, called the *standard form* of I, where $I_1, \ldots, I_s$ are nonsingular k-partitions such that $I^-_{a+1} - I^+_a > 3$, $1 \leqslant a \leqslant s - 1$, and the following two conditions are satisfied:

(1) If $I^- \leqslant 2k$, then I_1 is a main subpartition of I of maximum possible length. It is called the *main component* of I.

(2) The partitions I_a different from the main one are dense. They are called the *dense components* of I.
The parts I_r^- of the dense components I_r are called the *leading parts* of I. The number of leading parts is called the *index* of I and is denoted by $\operatorname{ind}_k I$.

For example, the standard form of $I = [2, 5, 9, 14, 17]$ is $[2, 5] \sqcup [9] \sqcup [14, 17]$ if I is treated as a 1-partition and $[2, 5, 9] \sqcup [14, 17]$ if I is treated as a 2-partition. Hence $\operatorname{ind}_1 I = 2$ and $\operatorname{ind}_2 I = 1$.

Definition 1.12. A *marked k-partition* is a pair $\{k; [I; J]\}$, where $[I; J] \in \mathbf{D}$, I is a k-partition, and the partition $I \setminus J$ is strict.

Definition 1.13. A marked k-partition $[I; J]$ is said to be *nonsingular* if I is a nonsingular k-partition and J is a subset of the set of leading parts of I. Otherwise, $[I; J]$ is said to be *singular*.

We use the following notation:
$\mathrm{N}(k)$ is the set of nonsingular k-partitions.
$\mathbf{D}(k)$ is the set of marked k-partitions.
$\mathbf{N}(k)$ is the set of nonsingular marked k-partitions.
$\mathrm{M}(k)$ is the set of main k-partitions.
$\mathrm{M}_q(k)$ is the set of main k-partitions of length q.
As the subsets of the set $\mathbf{D}$, all these sets inherit the partial order $\trianglelefteq$.

2. Filtering Basis Theorem and the Cohomology of the Algebras $L(k)$

Let $\{C_*(k), d\}$ be the exterior complex of the Lie algebra $L(k)$, where $C_*(k) = \bigoplus_{q \geqslant 0} C_q(k)$. Its space $C_q(k) = \Lambda^q(L(k))$ of q-dimensional chains consists of finite linear combinations of the basic vectors $e_I = e_{i_1} \wedge \cdots \wedge e_{i_q}$, called *$k$-monomials*, where $I = [i_1, \ldots, i_q]$ is a strict k-partition. The action of the boundary operator d is defined as

$$d_k(e_I) = \sum_{1 \leqslant s < t \leqslant q} (-1)^{s+t-1} (i_t - i_s)_h e_{i_s + i_t} \wedge e_{i_1} \wedge \cdots \wedge \widehat{e}_{i_s} \wedge \cdots \wedge \widehat{e}_{i_t} \wedge \cdots \wedge e_{i_q}. \tag{6}$$

Let $C_*^{(n)}(k)$ be the space spanned by the k-monomials e_I with $\|I\| = n$. Then $C_*(k) = \bigoplus_{n \geqslant 0} C_*^{(n)}(k)$ (a direct sum of complexes). We define the space of q-dimensional cochains as $C^q(k) = \bigoplus_{n \geqslant 0} C^q_{(n)}(k)$, where $C^q_{(n)}(k) = \operatorname{Hom}_{\mathbb{Q}}(C_q^{(n)}(k), \mathbb{Q})$.

Let us introduce an inner product in $C_*(k)$ by setting $\langle e_{I_1}, e_{I_2} \rangle = \delta_{I_1, I_2}$. It defines isomorphisms $C_q^{(n)}(k) \cong C^q_{(n)}(k)$, which allow us to treat the chains of the algebra $L(k)$ as cochains. Let δ_k be the (coboundary) operator on $C_*(k) = C^*(k)$ dual to d_k with respect to the inner product. The definitions imply the formulas

$$\delta_k(e_i) = \sum_{a+b=i;\, k \leqslant a < b} (b-a)_h e_a \wedge e_b, \tag{7}$$

$$\delta_k(e_{i_1} \wedge \cdots \wedge e_{i_q}) = \sum_{1 \leqslant a \leqslant q} (-1)^{a-1} e_{i_1} \wedge \cdots \wedge \delta_k(e_{i_a}) \wedge \cdots \wedge e_{i_q}. \tag{8}$$

The homology of the complexes $\{C_*(k), d_k\}$ and $\{C^*(k), \delta_k\}$ is called the *homology* and *cohomology*, respectively, of the algebra $L(k)$ and is denoted by $H_*(k)$ and $H^*(k)$.

Definition 2.1. Given $[I; J] \in \mathbf{D}(k)$, $I = [i_1, \ldots, i_q]$, define $e_{[I;J]} = e_{(i_1)} \wedge \cdots \wedge e_{(i_q)}$, where

$$e_{(i_a)} = \begin{cases} e_{i_a} & \text{if } i_a \notin J, \\ \delta_k(e_{i_a}); & \text{if } i_a \in J. \end{cases}$$

A cochain of the form $e_{[I;J]}$ is called an *$e(k)$-monomial*. It is called an *$\varepsilon(k)$-monomial* if $[I; J] \in \mathbf{N}(k)$ and a *singular $e(k)$-monomial* otherwise. For an $\varepsilon(k)$-monomial $e_{[I;J]}$, the notation $\varepsilon_{[I;J]}$ is used as well.

Keeping in mind the correspondence between marked k-partitions and $e(k)$-monomials, we shall apply the notions related to such partitions (degree, length, etc.) to $e(k)$-monomials. Note that $e_{[I;J]} = 0$ may be zero for a nonempty $[I;J]$. For example, $e_2 \wedge \delta_1 e_3 = 0$. This motivates the following definition.

Definition 2.2. We write $e_{[I;J]} \lhd e_{[I';J']}$ if $e_{[I';J']} \neq 0$ and ether $e_{[I;J]} = 0$ or $[I;J] \lhd [I';J']$.

Corollary 1.7 implies the following assertion.

Lemma 2.3. *If $e_{[I;J]} \trianglelefteqslant e_{[I';J']}$ and $e_{[I_1;J_1]} \lhd e_{[I_1';J_1']}$, then $e_{[I;J]} \wedge e_{[I_1;J_1]} \lhd e_{[I';J']} \wedge e_{[I_1';J_1']}$.*

Theorem 2.4. *The set of $\varepsilon(k)$-monomials is a basis of the space $C^*(k)$ and has the property that*

$$e_{[I;J]} = \sum_{[I';J'] \lhd [I;J]} \alpha_{[I';J']} \varepsilon_{[I';J']} \quad \text{if } [I;J] \text{ is a singular marked } k\text{-partition}, \tag{9}$$

$$\delta_k(\varepsilon_I) = \alpha_I \varepsilon_{[I;I^-]} + \sum_{[I';J'],\, I' \lhd I} \alpha_{[I';J']} \varepsilon_{[I';J']} \qquad (\alpha_I \neq 0) \quad \text{if } I \text{ is a dense } k\text{-partition}. \tag{10}$$

Formula (9) implies that for a dense or main k-partition I we have

$$\delta_k(\varepsilon_{[I;I^-]}) = \sum_{[I';J'],\, I' \lhd I} \alpha_{[I';J']} \varepsilon_{[I';J']} \quad \text{or} \quad \delta_k(\varepsilon_I) = \sum_{[I';J'],\, I' \lhd I,\, \mathrm{ind}_k I' > 0} \alpha_{[I';J']} \varepsilon_{[I';J']}, \tag{11}$$

respectively.

Let us apply Theorem 2.4 to compute $H^*(k) = H^*(L(k);\mathbb{Q})$. Let $\mathrm{N}_0(k) = \mathrm{N}(k)$ and

$$\mathrm{N}_{t+1}(k) = \mathrm{N}_t(k) \setminus \{\text{the set of maximal elements in } \mathrm{N}_t(k) \text{ with respect to } \trianglelefteqslant\},$$

where $t \in \mathbb{Z}_{\geqslant 0}$. Let $\pi\colon \mathbf{N}(k) \to \mathrm{N}(k)$ be the projection defined by $\pi[I;J] = I$, and let $E_t(k)$ be the linear span of the set of $\varepsilon(k)$-monomials $\varepsilon_{[I;J]}$, where $[I;J] \in \pi^{-1}(\mathrm{N}_t(k))$. By Theorem 2.4, $C^*(k) = E_0(k)$. Hence we obtain a filtration

$$C^*(k) = E_0(k) \supset E_1(k) \supset E_2(k) \supset \cdots \tag{12}$$

of vector spaces. Let $\varepsilon_{[I;J]} \in E_t(k)$, and let $I_1 \sqcup \cdots \sqcup I_m$ be the standard form of the k-partition I. By formula (8),

$$\begin{aligned} \delta_k(\varepsilon_{[I;J]}) &= \delta_k(\varepsilon_{[I_1;I_1 \cap J]} \wedge \cdots \wedge \varepsilon_{[I_s;I_s \cap J]}) \\ &= \sum_{1 \leqslant a \leqslant s} (-1)^{\eta_a(I;J)} \varepsilon_{[I_1;I_1 \cap J]} \wedge \cdots \wedge \delta_k \varepsilon_{[I_a;I_a \cap J]} \wedge \cdots \wedge \varepsilon_{[I_s;I_s \cap J]}, \end{aligned} \tag{13}$$

where $\eta_1(I;J) = 1$ and $\eta_a(I;J) = |I_1| + \cdots + |I_{a-1}| + |I_1 \cap J| + \cdots + |I_{a-1} \cap J|$ for $a > 1$. Formulas (10) and (11) imply that $\delta_k E_t(k) \subset E_t(k)$. Therefore, (12) is a filtration of the complex $\{C^*(k), \delta_k\}$.

Let $\{\mathscr{E}_r(k), \partial_r\}$ be the rth page of the corresponding spectral sequence. The image of an $\varepsilon(k)$-monomial $\varepsilon_{[I;J]}$ under the natural projection $C^*(k) \to \mathscr{E}_0(k)$ will be denoted by the same symbol. The differential ∂_0 acts by formula (13) with δ_k replaced by ∂_0.

For $I_a \in \mathrm{M}(k)$, it follows from the second formula in (11) and Lemma 2.3 that the corresponding term on the right-hand side in (13) belongs to $E_{t+1}(k)$. The same is true for $I_a^- \in J$, as follows from the first formula in (11) and Lemma 2.3. In particular, for a main or dense k-partition I we have

$$\partial_0 \varepsilon_{[I;J]} = \begin{cases} 0 & \text{if } |J| = 1 \text{ or } \mathrm{ind}_k I = 0, \\ \alpha_I \varepsilon_{[I;I^-]} \neq 0 & \text{if } |J| = 0 \text{ and } \mathrm{ind}_k I = 1. \end{cases} \tag{14}$$

Let $C(I) \subset \mathscr{E}_0(k)$ be the space spanned by all chains $\varepsilon_{[I;J]}$ with I fixed. Then formula (14) shows that $\mathscr{E}_0(k) = \bigoplus_{I \in \mathrm{N}(k)} C(I)$ (a direct sum of complexes).

It follows from (9), (10), and the definition of spectral sequence that $H^*(C(I)) \cong \mathbb{Q}\,\varepsilon_I$ if I is a main k-partition and $H^*(C(I)) = 0$ if I is a dense k-partition.

Formula (13) shows that $C(I) = \bigotimes_{1\leqslant s\leqslant m} C(I_s)$ is a tensor product of complexes. Therefore, by the Künneth formula,

$$H^*(C(I)) \cong \begin{cases} 0 & \text{if } \operatorname{ind}_k I \neq 0, \\ \mathbb{Q}\,\varepsilon_I & \text{if } \operatorname{ind}_k I = 0. \end{cases}$$

Thus, $\mathscr{E}_1(k) = \bigoplus_{I\in \mathrm{M}_q(k)} \mathbb{Q}\,\varepsilon_I$. Now it follows from (14) that $\partial_1 = \partial_2 = \dots = 0$. Hence $\mathscr{E}_\infty(k) = \mathscr{E}_1(k)$. The calculation is complete.

Theorem 2.5. *For any $I \in \mathrm{M}_q(k)$, there exists a cocycle*

$$\mathscr{C}_I = \varepsilon_I + \sum_{[I';J']\lhd I,\, \operatorname{ind}_k I'>0} \alpha_{[I';J']}\varepsilon_{[I';J']} \in C^q(k) \tag{15}$$

representing a nonzero class $\mathscr{C}_I \in H^q(k)$. The set of homology classes of the cocycles $\mathscr{C}_I$ is a basis of the space $H^q(k)$.

Remark 2.6. Theorem 2.5 shows that formula (3) is equivalent to the formula (see [7])

$$|\mathrm{M}_q(k)| = \binom{q+k-1}{k-1} + \binom{q+k-2}{k-1},$$

which can readily be verified by induction on k.

Remark 2.7. From Theorems 2.4 and 2.5, one can get some information on the multiplication induced in $H^*(k)$ by the exterior multiplication of cochains. Note that $\psi(k,|I|) \leqslant \|I\| \leqslant \psi(2k,|I|)$ for $I \in \mathrm{M}_q(k)$ and $\psi(k,q) = kq + 3q(q-1)/2$.

Assume that $I_1, I_2 \in \mathrm{M}(k)$ and $\mathscr{C}_{I_1} \wedge \mathscr{C}_{I_2}$ represents a nonzero class $\mathscr{C}_{I_1} \cdot \mathscr{C}_{I_2} \in H^*(k)$. Then, by formulas (15) and (9) and Lemma 2.3, there exists an $I \in \mathrm{M}_{|I_1|+|I_2|}(k)$ with $I \trianglelefteq I_1 \sqcup I_2$. Therefore, $\psi(k, |I_1|+|I_2|) \leqslant \|I_1\| + \|I_2\| \leqslant \psi(2k,|I_1|) + \psi(2k,|I_2|)$. This implies the condition $3|I_1||I_2| \leqslant k(|I_1|+|I_2|)$ necessary for $\mathscr{C}_{I_1} \cdot \mathscr{C}_{I_2}$ to be nonzero. It follows from this condition that the multiplication in the cohomology of $L(1)$ and $L(2)$ is trivial.

If $I_1, I_2, I_1 \sqcup I_2 \in \mathrm{M}(k)$, then $\mathscr{C}_{I_1} \cdot \mathscr{C}_{I_2} = \operatorname{sg}(I_1,I_2)\mathscr{C}_{I_1\sqcup I_2}$, where $\operatorname{sg}(I_1,I_2)$ is the sign of the permutation $\binom{I_1\sqcup I_2}{I_1,I_2}$. Hence the multiplication in $H^*(k)$ is nontrivial for $k>2$. Theorems 2.4 and 2.5 reduce computing the multiplication in $H^*(k)$ to computing the coefficients α_I in the expansion

$$\varepsilon_{I_1} \wedge \varepsilon_{I_2} = \sum_{I\lhd I_1\sqcup I_2} \alpha_I \varepsilon_I + \sum_{[I';J']\lhd I_1\sqcup I_2,\, \operatorname{ind}_k I'>0} \beta_{[I';J']}\varepsilon_{[I';J']},$$

where $I_1, I_2, I \in \mathrm{M}(k)$. To derive a general formula for these coefficients is apparently a difficult combinatorial problem.

3. Proof of (Main) Theorem 2.4

Lemma 3.1. *In the complex $C^*(k)$, we have*

$$\sum_{a+b=n} (a)_h\, e_a \wedge \delta_k(e_b) = 0, \tag{16}$$

$$\sum_{a+b=n} \delta_k(e_a) \wedge \delta_k(e_b) = 0, \qquad \sum_{a+b=n} (b-a)_h^2\, \delta_k(e_a) \wedge \delta_k(e_b) = 0. \tag{17}$$

If either $h=1$ or $h\neq 1$ and $n \not\equiv 0 \mod 3$, then

$$\sum_{a+b=n} e_a \wedge \delta_k(e_b) = 0, \qquad \sum_{a+b=n} (a)_h (b-a)_h^2\, e_a \wedge \delta_k(e_b) = 0. \tag{18}$$

If $h \neq 1$ and $n \equiv 0 \mod 3$, then

$$\sum_{a+b=n} (n-3a)\, e_a \wedge \delta_k(e_b) = 0, \qquad \sum_{a+b=n} (2-3(a)_h^2)\, e_a \wedge \delta_k(e_b) = 0. \tag{19}$$

Proof. The relation $\sum_{a+b=n} f(a,b)\, e_a \wedge \delta_k(e_b) = 0$ is equivalent to the identity

$$(y-z)_h\, f(x, y+z) - (x-z)_h\, f(y, x+z) + (x-y)_h\, f(z, x+y) = 0.$$

A straightforward verification shows that this identity holds for (16), (18), and (19). By applying δ_k to (18) or (19), we obtain (17). □

Definition 3.2. An $e(k)$-monomial $e_{[I;J]}$ is said to be *good* if the k-partition $[I;J]$ is nonsingular as a marked 1-partition and *bad* otherwise.

Lemma 3.3. *For any bad $e(k)$-monomial $e_{[I;J]} \in C^*(k)$, there exists a decomposition*

$$e_{[I;J]} = \sum_{[I';J'] \lhd [I;J]} \alpha_{[I';J']} e_{[I';J']}, \tag{20}$$

where all $e(k)$-monomials $e_{[I';J']}$ are good.

Proof. Obviously, an $e(k)$-monomial is bad if and only if it contains length 2 submonomials from the following list, where $\alpha = 0, 1$:

$$x(i,\alpha) = e_{i-1+\alpha} \wedge e_{i+1},$$
$$y_1(i,\alpha) = e_i \wedge \delta_k e_{i+\alpha}, \quad y_2(i,\alpha) = \delta_k e_{i-1+\alpha} \wedge e_{i+1}, \quad y_3(i,\alpha) = e_{i-1} \wedge \delta_k e_{i+1+\alpha},$$
$$z_1(i,\alpha) = \delta_k e_i \wedge \delta_k e_{i+\alpha}, \quad z_2(i,\alpha) = \delta_k e_{i-1} \wedge \delta_k e_{i+1+\alpha}.$$

Let $|I| = 2$. Consider (7) as an equation for the unknown $x(i,\alpha)$; identities (16) and (18) for $h = 1$ or $h \neq 1$ and $n \not\equiv 0 \mod 3$, or identities (16) and (19) for $h \neq 1$ and $n \equiv 0 \mod 3$ as a system of equations for the unknowns $y_1(i,\alpha)$, $y_2(i,\alpha)$, and $y_3(i,\alpha)$; and finally (17) as a system of equations for the unknowns $z_1(i,\alpha)$ and $z_2(i,\alpha)$. A straightforward verification shows that these systems of linear equations are nonsingular. Their solutions give the decomposition (20) for bad $e(k)$-monomials of length 2.

For $|I| > 2$, let us replace an arbitrary (e.g., the leftmost) bad $e(k)$-submonomial of length 2 in $e_{[I;J]}$ by a linear combination of good ones. Then $e_{[I;J]}$ will be expressed as a linear combination of $e(k)$-monomials each of which is less than $e_{[I;J]}$ by Lemma 2.3. Let us apply the same procedure to each bad term in the resulting sum, etc. Since the number of marked partitions $[I';J'] \lhd [I;J]$ is finite, we arrive at the decomposition (20) in finitely many steps. □

Definition 3.4. For $c = \sum \alpha_{[I;J]} e_{[I;J]} \in C^*(k)$, we write $c \approx 0$ if $c = \sum \alpha_{[I';J']} e_{[I';J']}$ and $[I';J'] \lhd [I;J]$ for all pairs $[I;J]$ and $[I';J']$ with $\alpha_{[I;J]}\alpha_{[I';J']} \neq 0$. For $c_1, c_2 \in C^*(k)$, we write $c_1 \approx c_2$ if $c_1 - c_2 \approx 0$.

Lemma 3.5. *Let I be a dense k-partition. Then $\delta_k(\varepsilon_I) \approx \alpha_I \varepsilon_{[I;I^-]}$, where $\alpha_I \neq 0$.*

Proof. By applying the algorithm of Lemma 3.3 to I, we obtain

$$e_{[I;I^+]} \approx \begin{cases} (-1)^{|I|-1} I^+/I^-\, \varepsilon_{[I;I^-]} & \text{if } h = 1 \text{ or } h \neq 1 \text{ and } n \not\equiv 0 \mod 3, \\ (-1)^{|I|-1}\, \varepsilon_{[I;I^-]} & \text{if } h \neq 1 \text{ and } n \equiv 0 \mod 3. \end{cases} \tag{21}$$

If $I = [i_1, \dots, i_q]$ and $I_r = [i_1, \dots, i_r]$, then

$$\delta_k(\varepsilon_I) = \sum_{1 \leqslant r \leqslant q} (-1)^{r-1} e_{[I_r; I_r^+]} \wedge \varepsilon_{I \setminus I_r}.$$

By substituting the expressions (21) for $e_{[I_r;I_r^+]}$ into this formula, we obtain

$$\delta_k(\varepsilon_I) \approx \begin{cases} \|I\|/I^-\, \varepsilon_{[I;I^-]}, & \text{if } h = 1 \text{ or } h \neq 1 \text{ and } n \not\equiv 0 \mod 3, \\ |I|\, \varepsilon_{[I;I^-]}, & \text{if } h \neq 1 \text{ and } n \equiv 0 \mod 3. \end{cases}$$

This completes the proof. □

The sets of good $e(1)$-monomials and $\varepsilon(1)$-monomials coincide. Hence Lemmas 3.3 and 3.5 imply that the set of $\varepsilon(1)$-monomials of degree n is a linear system of generators of the vector space $C_*^{(n)}(1)$, for which formulas (9) and (10) hold. To finish the proof of Theorem 2.4 for $k=1$, it suffices to show that the $\varepsilon(1)$-monomials are linearly independent.

Let $\mathbf{N}_{l,m}^{(n)} = \{[I;J] \in \mathbf{N}(1) : \|I\| = n,\ |I| = l,\ |J| = m\}$. The number of $\varepsilon(1)$-monomials of dimension q and degree n is equal to the cardinality of the set $\mathbf{N}_q^{(n)} = \bigsqcup_{l+m=q} \mathbf{N}_{l,m}^{(n)}$. On the other hand, $\dim C^q_{(n)}(1) = |\mathrm{D}_q^{(n)}|$, where $\mathrm{D}_q^{(n)}$ is the set of strict 1-partitions of length q and degree n. Therefore, the linear independence of $\varepsilon(1)$-monomials is a consequence of the following claim:

Theorem 3.6. $|\mathrm{D}_q^{(n)}| = |\mathbf{N}_q^{(n)}|$.

It suffices to show that

Lemma 3.7. *The claim of Theorem* 3.6 *and identity* (4) *are equivalent.*

Proof. Let $A_l(x,t) = \sum_{n=1}^\infty \sum_{m=0}^l |\mathbf{N}_{l,m}^{(n)}| t^m x^n$. One can readily see that Theorem 3.6 is equivalent to the identity

$$\prod_{i=1}^\infty (1+tx^i) = 1 + \sum_{n=1}^\infty \sum_{q=1}^\infty |\mathbf{N}_q^{(n)}(1)|\, t^q x^n = 1 + \sum_{l=1}^\infty A_l(x,t)\, t^l.$$

Now it suffices to verify that

$$A_l(x,t) = x^{(3l^2-l)/2} \frac{(1+tx)\cdots(1+tx^{l-1})(1+tx^{2l})}{(1-x)\cdots(1-x^{l-1})(1-x^l)}\,.$$

The property of a 1-partition $[i_1,\dots,i_l]$ to be nonsingular is equivalent to the following property of the dual partition (see [10]): it is a partition in which $i_1 \geqslant 1$ parts are equal to l, $i_2 - i_1 \geqslant 3$ parts are equal to $l-1, \dots, i_l - i_{l-1} \geqslant 3$ parts are equal to 1. If either $i_1 \geqslant 3$ or $i_a - i_{a-1} > 3$ for some a, where $2 \leqslant a \leqslant q$, then the corresponding part (i_1 or i_a) of the partition can be either marked (the coefficient t) or not (the coefficient 1). Thus,

$$A_{1,l}(x,t) = \left(x^l + x^{2l} + (1+t)\sum_{r=3}^\infty x^{rl}\right)\left(x^{3(l-1)} + (1+t)\sum_{r=4}^\infty x^{r(l-1)}\right)\cdots\left(x^3 + (1+t)\sum_{r=4}^\infty x^r\right).$$

By summing the geometric progressions in the parentheses, we arrive at the desired formula. $\square$

The proof of Theorem 2.4 for $k=1$ is complete. For $k>1$, we shall prove it by induction on k. First, we introduce two useful operations on marked partitions.

Definition 3.8. Let $\sigma[i_1,\dots,i_q] = [i_1+1,\dots,i_q+1]$. Define a mapping $\sigma\colon \mathbf{D}(k) \to \mathbf{D}(k+1)$ by the formula $\sigma[I;J] = [\sigma(I\setminus J) \sqcup \sigma^2(J); \sigma^2(J)]$.

Definition 3.9. Let $[I;J] \in \mathbf{N}(k)$, let $I_1 \sqcup \cdots \sqcup I_m$ be the standard form of the k-partition I, and let $J = [I_{a_1}^-,\dots,I_{a_r}^-]$. Define $\tau[I;J] = [I;J^+] \in \mathbf{D}(k)$, where $J^+ = [I_{a_1}^+,\dots,I_{a_r}^+]$.

The definitions readily imply that

Lemma 3.10. *The composition $\sigma\tau$ induces a bijective mapping $\sigma\tau\colon \mathbf{N}(k) \to \mathbf{N}(k+1)$.*

Assume that Theorem 2.4 is true for $\varepsilon(k-1)$-monomials. Then the set of monomials $e_{\tau[I;J]}$, where $[I;J] \in \mathbf{N}(k-1)$, is a basis of $C^*(k-1)$, and

$$\varepsilon_{[I;J]} = \beta_{[I;J]}\, e_{\tau[I;J]} + \sum_{[I';J'] \lhd [I;J]} \beta_{[I';J']}\, e_{\tau[I';J']}, \qquad \text{where } \beta_{[I;J]} \neq 0. \tag{22}$$

Indeed, formula (21) and Lemma 2.3 show that

$$e_{\tau[I;J]} \approx \gamma(I_{a_1})\cdots\gamma(I_{a_1})\,\varepsilon_{[I;J]} \neq 0,$$

where $I_1 \sqcup \cdots \sqcup I_m$ is the standard form of the $(k-1)$-partition I, $J = [I_{a_1}^-,\dots,I_{a_r}^-]$, and $\gamma(I_a)$ is the coefficient of $\varepsilon_{[I_a;I_a^-]}$ in formula (21). Hence, under the linear ordering of $\mathbf{N}(k-1)$ compatible with $\trianglelefteq$, the subset of $e(k-1)$-monomials $e_{\tau[I;J]}$ of fixed degree can be expressed via the basis of $\varepsilon(k-1)$-monomials of the same degree by an invertible upper triangular matrix. By inverting this matrix, we see that the $e(k-1)$-monomials $e_{\tau[I;J]}$ form a basis of the space $C^*(k-1)$ as well and that formula (22) holds for this basis, since the inequality $\tau[I';J'] \lhd [I;J]$ implies that $[I';J'] \lhd [I;J]$.

Let $\sigma\colon C^*(k-1) \to C^*(k)$ be the linear isomorphism defined by the formula $\sigma(e_I) = e_{\sigma(I)}$. Since $\sigma\delta_k(e_i) = \delta_{k+1}(e_{i+2})$, we have $\sigma(e_{[I;J]}) = e_{\sigma[I;J]}$. By applying σ to both sides of (22), we obtain

$$\sigma(\varepsilon_{[I;J]}) = \beta_{[I;J]}\,\varepsilon_{\sigma\tau[I;J]} + \sum_{[I';J']\lhd[I;J]} \beta_{[I';J']}\,\varepsilon_{\sigma\tau[I';J']}, \quad \text{where } \beta_{[I;J]} \neq 0, \tag{23}$$

since $e_{\sigma\tau[I';J']} = \varepsilon_{\sigma\tau[I';J']}$ by Lemma 3.10.

By the induction assumption, the set of chains $\sigma(\varepsilon_{[I;J]})$ is a basis of $C^*(k)$, since σ is an isomorphism. On the other hand, the set of chains $\varepsilon_{\sigma\tau[I;J]}$ coincides with the set of $\varepsilon(k)$-monomials by Lemma 3.10. As earlier, Eq. (23) implies that these sets are related by an invertible triangular transformation. Thus, the $\varepsilon(k)$-monomials form a basis of $C^*(k)$ as well. Now formula (10) follows from Lemma 3.5. It remains to prove formula (9).

Note that a good $e(k)$-monomial $e_{[I;J]}$ is not an $\varepsilon(k)$-monomial if and only if the main component of the nonsingular k-partition I is not empty and contains marked parts. If such $e(k)$-monomials are absent in the decomposition (20), then the decomposition coincides with (9).

An algorithm similar to that used in the proof of Lemma 3.3 shows that it suffices to consider the case in which $I = [i_1, \dots, i_q] \in \mathrm{M}(k)$, $J = [i_q]$, and $i_q - i_{q-1} > 3$. Then $\sigma^{-1}[I;I^+] = [I_1;I_1^+]$, where $I_1 \in \mathrm{M}(k-1)$. By the induction assumption,

$$\sigma^{-1}(e_{[I;I^+]}) = \sum_{[I';J']\lhd\sigma^{-1}[I;I^+],\,[I';J']\in\mathbf{N}(k-1)} \alpha_{[I';J']}\,e_{\tau[I';J']}.$$

An application of the operator σ to this formula gives the decomposition (9) for $e_{[I;I^+]}$, since the following claim is true:

Lemma 3.11. *Let $I \in \mathrm{M}(k)$, $[I';J'] \in \mathbf{N}(k)$, and $[I';J'] \lhd [I;I^+]$. Then $\sigma\tau[I';J'] \lhd \sigma[I;I^+]$.*

Proof. The claim is nontrivial only for $|I'| = |I|$. Let $I' = I_1' \sqcup \dots \sqcup I_s'$ be the standard form, and let $J' = [I_l'^-]$. Since $I' \lhd I$ and I' is nonsingular, it follows that $I_1' \neq \varnothing$ is a main k-partition and $I_2' \neq \varnothing$. Let $|I| = q$ and $I_l'^+ = i_p'$. For $p = q$, the claim is obvious. Assume that $p < q$.

It suffices to show that inequalities (5) are strict for all $r \geqslant p$. Indeed, if $r \geqslant p$, then $\sigma\tau[I';J'] \lhd \sigma[I;I^+]$. But $\sigma[I;I^+] \notin \mathbf{N}(k+1)$, while $\sigma\tau[I';J'] \in \mathbf{N}(k+1)$ by Lemma 3.10. Hence $\sigma\tau[I';J'] \lhd \sigma[I;I^+]$.

For $p \leqslant r \leqslant q$, the definitions imply that

$$\begin{aligned} &i_r < 2k + 3(r-1) \leqslant i_r' \quad \text{if } i_1 = k, \\ &i_r \leqslant 2k + 3(r-1) < i_r' \quad \text{if } i_1 > k \text{ and } i_1' > k, \\ &i_r \leqslant 2k + 3(r-1) \leqslant i_r' \quad \text{if } i_1 > k \text{ and } i_1' = k. \end{aligned} \tag{24}$$

Assume first that $i_r' > i_r$ for all $r \geqslant p$. Then the minimum a such that $p \leqslant a \leqslant q$ and $i_1' + \dots + i_a' = i_1 + \dots + i_a$ is equal to q. Hence inequalities (5) are strict for $r \geqslant p$.

Suppose that there exists an $r \geqslant p$ with $i_r' = i_r$, and let r_0 be the maximum r with this property. It follows from (24) that $i_{r_0}' = i_{r_0} = 2k + 3(r_0 - 1)$. Hence $l = 2$ and $p = r_0$. Since $i_p' = i_p$ and $i_r' > i_r$ for $r > p$, we can argue as before to see that inequalities (5) are strict for $r \geqslant p$. $\square$

4. Stable Filtration of the Complex $C_*(k)$

In this section, we treat $e(k)$-monomials as vectors in the space $C_*(k)$.

Definition 4.1. Let $\mathrm{St}_0(k) = 0$ and

$$\mathrm{St}_m(k) = \{c \in C_*(k) : d_{k+r}\sigma^r(c) \in \mathrm{St}_{m-1}(k+r) \text{ for all } r \geqslant 0\}$$

for any integer $m > 0$. The vectors in the space $\mathrm{St}_m(k)$ are called *m-stable chains*, and the filtration

$$0 = \mathrm{St}_0(k) \subset \mathrm{St}_1(k) \subset \mathrm{St}_2(k) \subset \cdots \subset \mathrm{St}_\infty(k) = C_*(k)$$

is called the *stable filtration* of $C_*(k)$. A *stable cycle* is a 1-stable chain (see [7]).

Definition 4.2. Let $c = \sum \alpha_{[I;J]} \varepsilon_{[I;J]}$ be the expansion of $c \in C_*(k)$ in the basis of $\varepsilon(k)$-monomials. Let $\mathrm{ht}_k(c) = \min\{|J| : \alpha_{[I;J]} \neq 0\}$ and $\mathrm{ht}_k(0) = \infty$. Denote by $\mathrm{E}_m(k) = \mathrm{E}_m(L(k))$ the space of chains $c \in C_*(k)$ with $\mathrm{ht}_k(c) \geqslant m$. The filtration

$$C_*(k) = \mathrm{E}_0(k) \supset \mathrm{E}_1(k) \supset \mathrm{E}_2(k) \supset \cdots \supset \mathrm{E}_\infty(k) = 0$$

is called the *ε-filtration* of $C_*(k)$.

Theorem 4.3. $\mathrm{St}_m(k) = \widehat{\mathrm{E}}_m(k) := \{c \in C^(k) : \langle c, \mathrm{E}_m(k)\rangle = 0\}$.

Lemma 4.4. *If $m \geqslant 1$, then for any $c \in \mathrm{E}_m(k)$ and $r \gg 0$ there exists a chain $c_r \in \mathrm{E}_{m-1}(k+r)$ such that $\sigma^r(c) = \delta_{k+r}(c_r)$.*

Proof. We say that a partition of the form $[I;J] = [i_1, \ldots, i_p, \underline{i_{p+1}}, \ldots, \underline{i_q}] \in \mathbf{N}(k)$, where $i_p \leqslant 2k$, is *stable*. For stable $[I;J]$, the following formula proves the lemma for $c = \varepsilon_{[I;J]}$:

$$\varepsilon_{[I;J]} = \varepsilon_{[i_1,\ldots,i_p]} \wedge \varepsilon_{[\underline{i_{p+1}},\ldots,\underline{i_q}]} = \delta_k(\varepsilon_{[i_1,\ldots,i_p]} \wedge \varepsilon_{[i_{p+1},\underline{i_{p+2}},\ldots,\underline{i_q}]}).$$

For arbitrary $[I';J'] \in \mathbf{N}(k)$, there exists an $r(I';J') \gg 0$ such that $(\sigma\tau)^{r(I';J')}[I';J']$ is stable. But then there exists an $r(I;J)$ such that $(\sigma\tau)^{r(I;J)}[I';J']$ is stable for all $[I';J'] \in \mathbf{N}(k)$ with $[I';J'] \trianglelefteq [I;J]$, because the number of such $[I';J']$ is finite.

It follows from (23) by induction on r that

$$\sigma^r(\varepsilon_{[I;J]}) = \beta_{r,[I;J]}\, \varepsilon_{(\sigma\tau)^r[I;J]} + \sum_{[I';J'] \triangleleft [I;J]} \beta_{r,[I';J']}\, \varepsilon_{(\sigma\tau)^r[I';J']}, \quad \text{where } \beta_{r,[I;J]} \neq 0, \tag{25}$$

for $r > 0$. Let $c = \sum_{[I;J]} \alpha_{[I;J]} \varepsilon_{[I;J]}$ and $r(c) = \max\{r(I;J) : \alpha_{[I;J]} \neq 0\}$. By applying $\sigma^{r(c)}$ to c, we obtain Lemma 4.4 from formula (25) and the already considered case of stable $[I;J]$. $\square$

Proof of Theorem 4.3. We argue by induction on m. For $m = 0$, the claim is obvious. Assume that $\mathrm{St}_p(k) = \widehat{\mathrm{E}}_p(k)$ for $p < m$.

First, let us show that the chain $c \in \widehat{\mathrm{E}}_m(k)$ is m-stable. It follows from formula (25) that $\sigma^r(\mathrm{E}_m(k)) = \mathrm{E}_m(k+r)$. Since $\langle \sigma^r(c_1), \sigma^r(c_2)\rangle = \langle c_1, c_2\rangle$ for arbitrary $c_1, c_2 \in C_*(k)$, we have

$$\langle \sigma^r(c), \mathrm{E}_m(k+r)\rangle = \langle \sigma^r(c), \sigma^r(\mathrm{E}_m(k))\rangle = \langle c, \mathrm{E}_m(k)\rangle = 0.$$

Hence $\sigma^r(c) \in \widehat{\mathrm{E}}_m(k+r)$. On the other hand, $\delta_{k+r}(\mathrm{E}_{m-1}(k+r)) \subset \mathrm{E}_m(k+r)$ by Theorem 2.4. Now the desired inclusion $d_{k+r}\sigma^r(c) \in \mathrm{St}_{m-1}(k) = \widehat{\mathrm{E}}_{m-1}(k+r)$ follows from the identity

$$\langle d_{k+r}\sigma^r(c), \mathrm{E}_{m-1}(k+r)\rangle = \langle \sigma^r(c), \delta_{k+r}(\mathrm{E}_{m-1}(k+r))\rangle = 0.$$

It remains to show that each m-stable chain belongs to $\widehat{\mathrm{E}}_m(k)$. Since $C_*(k) = \widehat{\mathrm{E}}_m(k) \oplus \mathrm{E}_{m+1}(k)$, it suffices to show that for each $c \in \mathrm{E}_{m+1}(k)$ there exists an $r \geqslant 0$ such that $d_{k+r}\sigma^r(c) \notin \widehat{\mathrm{E}}_{m-1}(k+r)$.

By Lemma 4.4, there exist r and $c_r \in \mathrm{E}_m(k+r)$ such that $\sigma^r(c) = \delta_{k+r}(c_r)$. Therefore,

$$\langle d_{k+r}\sigma^r(c), c_r\rangle = \langle \sigma^r(c), \delta_{k+r}(c_r)\rangle = \langle \sigma^r(c), \sigma^r(c)\rangle \neq 0.$$

Thus, $d_{k+r}\sigma^r(c) \notin \widehat{\mathrm{E}}_m(k+r)$. But then $d_{k+r}\sigma^r(c) \notin \widehat{\mathrm{E}}_{m-1}(k+r) \subset \widehat{\mathrm{E}}_m(k+r)$. $\square$

Corollary 4.5. *For any $[I;J] \in \mathbf{N}(k)$, denote by $\widehat{\varepsilon}_{[I;J]} \in C_*(k)$ the chain such that*

$$\langle \widehat{\varepsilon}_{[I;J]}, \varepsilon_{[I';J']}\rangle = \delta_{[I;J],[I';J']}$$

for all $[I';J'] \in \mathbf{N}(k)$. The set of chains $\widehat{\varepsilon}_{[I;J]}$ is a basis of $C_*(k)$. The chains $\widehat{\varepsilon}_{[I;J]}$ with $|J| \leqslant m-1$ form a basis of $\mathrm{St}_m(k)$. In particular, the chains $\widehat{\varepsilon}_I$ with $I \in \mathbf{N}(k)$ form a basis of the subspace of stable cycles in $C_*(k)$.

Proof. The claim on the basis follows from Theorem 2.4 and the nondegeneracy of the inner product. The second claim follows from Theorem 4.3. □

5. Stable Filtration and Skew-Symmetric Polynomials

Recall several facts on (skew)symmetric polynomials (see [10, Chap. I, Secs. 3, 5]). Let $\mathrm{Sym}_q[t]$ be the $\mathbb{Q}$-algebra of symmetric polynomials, and let $\mathrm{Alt}_q[t]$ be the vector space of skew-symmetric polynomials with rational coefficients in the variables $t_1, \ldots, t_q$. Set $\mathrm{Sym}_0[t] = \mathrm{Alt}_0[t] = \mathbb{Q}$. If there is no room for confusion, we write $f(t)$ instead of $f(t_1, \ldots, t_q)$. The polynomials

$$\Delta_I(t_1, \ldots, t_q) = \det(t_r^{i_m})_{(r,m=1,\ldots,q)}$$

corresponding to strict partitions $I = [i_1, \ldots, i_q]$ form a basis of $\mathrm{Alt}_q[t]$. The multiplication

$$\Delta_I(t_1, \ldots, t_q) \wedge \Delta_{I'}(t_1, \ldots, t_{q'}) = \mathrm{sg}(I, I')\, \Delta_{I \sqcup I'}(t_1, \ldots, t_{q+q'}),$$

where $\mathrm{sg}(I_1, I_2)$ is the sign of the permutation $\binom{I_1 \sqcup I_2}{I_1, I_2}$, equips $\mathrm{Alt}[t] = \bigoplus_{q=0}^{\infty} \mathrm{Alt}_q[t]$ with the structure of an anticommutative graded $\mathbb{Q}$-algebra.

The *Vandermonde determinant* is the polynomial $V_q(t) = \Delta_{\rho(q)}(t)$, where $\rho(q) = [0, 1, \ldots, q-1]$. The symmetric polynomial $S_I(t) = \Delta_{I+\rho(q)}(t)/V_q(t)$, where I is a partition with $|I| = q$, is called a *Schur polynomial*. The Schur polynomials form a basis in $\mathrm{Sym}_q[t]$. It is known that

$$S_{I_1}(t) S_{I_2}(t) = S_{I_1+I_2}(t) + \sum_{I \rhd I_1+I_2} \alpha_I S_I(t), \tag{26}$$

where $I_1 + I_2$ is the componentwise sum of partitions and $\alpha_I \in \mathbb{Z}$. One can readily show that

$$V_q^2(t) = S_{2\rho(q)}(t) + \sum_{I \rhd 2\rho(q)} \beta_I S_I(t). \tag{27}$$

We identify every chain $c = \sum_I r_I e_I \in C_q(k)$ with the polynomial $c(t) = \sum_I r_I \Delta_I(t)$.

Lemma 5.1. *Let $c \in C_q(L^{(h)}(k))$ and $q \geqslant 2$. Then*

$$d_k(c(t)) = \sum_{1 \leqslant a \leqslant q-1} (-1)^{a-1} c_a(t_1, \ldots, t_{q-1}; h), \tag{28}$$

where $c_a(t_1, \ldots, t_{q-1}; h) = (h^2 - h)^{-1} c(t_1, \ldots, t_{a-1}, ht_a, h^2 t_a, t_{a+1}, \ldots, t_{q-1})$.

Proof. It suffices to consider the case in which $c(t) = \Delta_I(t)$. For $q = 2$, formula (28) directly follows from (1). For $q > 2$, it follows by induction on q from the formula of the d_k-action on k-monomials. In the proof, it is convenient to use the formula

$$t_1^m \wedge c(t_1, \ldots, t_{q-1}) = \sum_{1 \leqslant r \leqslant q} (-1)^{p-1}\, t_r^m\, c(t_1, \ldots, \widehat{t_r}, \ldots, t_q),$$

where $c(t_1, \ldots, t_{q-1}) \in \mathrm{Alt}_{q-1}[t]$. We omit the corresponding routine computation. □

Let $\mathrm{Sym}^{(h)}_{m,q}[t] \subset \mathrm{Sym}_q[t]$ be the space of polynomials $p(t)$ such that

$$\mathrm{T}^{(h)}_{s,q}(p(t)) = p(ht_1, \overline{h}t_1, \ldots, ht_s, \overline{h}t_s, t_{s+1}, \ldots, t_{q-s}) = 0$$

for all s, $0 \leqslant s \leqslant m \leqslant \lfloor q/2 \rfloor$. For $c \in C_q(k)$, let $\widetilde{c}(t) \in \mathrm{Sym}_q[t]$ be the unique polynomial such that $c(t) = (t_1 \cdots t_q)^k V_q(t)\, \widetilde{c}(t)$.

Theorem 5.2. $c \in \mathrm{St}_{m,q}(k) = \mathrm{St}_m(k) \cap C_q(k)$ *if and only if* $\widetilde{c}(t) \in \mathrm{Sym}^{(h)}_{m,q}[t]$.

Proof. The operator σ acts on $c(t)$ as the multiplication by $t_1 \cdots t_q$. Therefore, it suffices to consider the case of $k = 1$. We argue by induction on m. For $m = 0$, the claim is obvious. Let $m \geqslant 1$. Since $V_q(t) = \prod_{q \geqslant i > j \geqslant 1} (t_i - t_j)$, we have

$$c_a(t_1, \ldots, t_{q-1}; h) = (t_1 \cdots t_{q-1})^k V_{q-1}(t)\, R_a(t; h)\, \varphi_a(t; h),$$

where $\varphi_a(t;h) = \varphi_a(t_1,\dots,t_{q-1};h) = \widetilde{c}(t_1,\dots,t_{a-1},ht_a,\overline{h}t_a,t_{a+1},\dots,t_{q-1})$ and

$$R_a(t;h) = R_a(t_1,\dots,t_{q-1};h) = t_a \prod_{1\leqslant i\leqslant q-1,\, i\neq a} \frac{(ht_a-t_i)(\overline{h}t_a-t_i)}{t_a-t_i}.$$

For $c \in \mathrm{St}_{m,q}(1)$, let us apply formula (28) to $\sigma^r(c) \in C_q(r+1)$. After cancelling the common factor $(t_1\cdots t_{q-1})^{r+1}V_{q-1}(t)$ we see by the induction assumption that $c \in \mathrm{St}_{m,q}(1)$ if and only if

$$t_1^{r-1}R_1(t;h)\,\varphi_1(t;h) + \cdots + t_{q-1}^{r-1}R_{q-1}(t;h)\,\varphi_{q-1}(t;h) = p_r(t;h) \in \mathrm{Sym}^{(h)}_{m-1,q-1}[t] \tag{29}$$

for every integer $r \geqslant 1$. For $r = 1,\dots,q-1$, we obtain a system of linear equations for the unknowns $X_a(t;h) = R_a(t;h)\varphi_a(t;h)$.

Let us apply $\mathrm{T}^{(h)}_{m-1,q-1}$ to (29). Since $\mathrm{T}^{(h)}_{m-1,q-1}(p_r(t;h)) = 0$, we obtain $\mathrm{T}^{(h)}_{m-1,q-1}(X_a(t;h)) = 0$ for $1 \leqslant a \leqslant 2(m-1)$, because $\mathrm{T}^{(h)}_{m-1,q-1}(R_a(t;h)) = 0$.

Therefore, for $2(m-1)+1 \leqslant r,\ a \leqslant q-1$ Eqs. (29) turn into a system of linear equations for the unknowns $\mathrm{T}^{(h)}_{m-1,q-1}(X_a(t;h))$. This system has only the zero solution, since its determinant $(t_{2m-1}\dots t_{q-m})^{2(m-1)}V_{q-2m+1}(t_{2m-1},\dots,t_{q-m})$ is nonzero. Since $\mathrm{T}^{(h)}_{m-1,q-1}(R_{2(m-1)+1}(t;h)) \neq 0$, we obtain $\mathrm{T}^{(h)}_{m-1,q-1}(\varphi_{2(m-1)+1}(t;h)) = \mathrm{T}^{(h)}_{m,q}(\widetilde{c}(t)) = 0$. $\square$

Corollary 5.3 (cf. [1]). *A chain $c \in C_q(L^{(h)}(k))$ is a stable cycle for $h = 1$ if and only if $c(t)$ is divisible by $V_q^3(t)$ and for $h \neq 1$ if and only if $c(t)$ is divisible by $V_q(t^3) = V_q(t_1^3\delta t_q^3)$.*

Proof. For $h = 1$, Theorem 5.2 implies that $\widetilde{c}(t)$ is divisible by $t_2 - t_1$ and hence, by symmetry, by $V_q^2(t)$. But then $c(t)$ is divisible by $V_q^3(t)$. For $h \neq 1$, Theorem 5.2 implies that $\widetilde{c}(t)$ is divisible by $(t_2 - ht_1)(t_2 - \overline{h}t_1)$. Then $c(t)$ is divisible by $(t_2-t_1)(t_2-ht_1)(t_2-\overline{h}t_1) = t_2^3 - t_1^3$ and hence by $V_q(t^3)$. The sufficiency follows from (28). $\square$

Corollary 5.4. *The polynomials $((t_1\cdots t_q)^k V_q(t))^{-1}\widehat{\varepsilon}_{[I;J]}(t)$, where $[I;J] \in \mathbf{N}(k)$, $|I|+|J| = q$, and $|J| = m-1$, form a basis of the space $\mathrm{Sym}^{(h)}_{m,q}[t]$.*

The dimension of the subspace of homogenous polynomials of degree n is equal to the number of partitions $[I;J] \in \mathbf{N}(1)$ with $|I|+|J| = q$, $|J| = m-1$, and $\|I\| = n + q(q+1)/2$.

Proof. The first claim readily follows from Theorem 5.2 and Corollary 4.5. The second is the special case of the first for $k = 1$. $\square$

6. Stable Cycles and the Homology of the Algebras $L(k)$

Theorem 6.1. *The homology classes of the stable cycles $\widehat{\varepsilon}_I$, where $I \in \mathrm{M}_q(k)$, form a basis of the space $H_q(L^{(h)}(k))$.*

Proof. By Theorem 2.5, for $I_0 \in \mathrm{M}_q(k)$ there exists a cocycle $\mathscr{C}_{I_0} \in C^q(k)$ such that $\langle \mathscr{C}_{I_0}, \widehat{\varepsilon}_I\rangle = \delta_{I_0,I}$ for each $I \in \mathrm{M}_q(k)$. Since $\widehat{\varepsilon}_I$ is a cycle for each $I \in \mathbf{N}(k)$, it follows that the chain $z = \sum_{I\in \mathrm{M}_q(k)} \beta_I \widehat{\varepsilon}_I \in C_q(k)$ represents a homology class in $H_q(k)$. If $z = d_k(x)$, then $\beta_{I_0} = \langle z, \mathscr{C}_{I_0}\rangle = \langle d_k(x), \mathscr{C}_{I_0}\rangle = \langle x, \delta_k(\mathscr{C}_{I_0})\rangle = 0$. Thus, each nonzero vector in the space generated by the cycles $\widehat{\varepsilon}_I$, where $I \in \mathrm{M}_q(k)$, represents a nonzero q-dimensional class in $H_q(k)$. Since the vectors $\widehat{\varepsilon}_I$ are linearly independent, this completes the proof, because $\dim H_q(k) = |\mathrm{M}_q(k)|$ by Theorem 2.5. $\square$

Theorem 6.2. *The homology classes of the stable cycles*

$$Z_I(t) = \begin{cases} S_{I-3\rho(q)}(t)\,V_q^3(t) & \text{if } h = 1,\\ S_{I-3\rho(q)}(t)\,V_q(t^3) & \text{if } h \neq 1,\end{cases}$$

where $I \in \mathrm{M}_q(k)$, form a basis of the space $H_q(L^{(h)}(k))$.

Proof. It suffices to prove the formula

$$Z_I(t) = \widehat{\varepsilon}_I(t) + \sum_{I'\triangleright I} \alpha_{I'}\widehat{\varepsilon}_{I'}(t). \tag{30}$$

Indeed, if $I \in \mathrm{M}(k)$ and $I' \rhd I$ is a nonsingular k-partition, then $I' \in \mathrm{M}(k)$. Therefore, if $I \in \mathrm{M}(k)$, then all I' in formula (30) belong to $\mathrm{M}(k)$, and the matrix of transition from the set of chains $Z_I(t)$ to the set of chains $\widehat{\varepsilon}_I(t)$ is upper triangular in the linear order on $\mathbf{D}(k)$ consistent with $\trianglelefteq$. By Theorem 6.1, the chains $\widehat{\varepsilon}_I$ form a basis of the homology space. Thus, the same is true for the set of chains $Z_I(t)$.

Note that $Z_I(t) = \Delta_I(t) + \sum_{I' \rhd I} \alpha_{I'} \Delta_{I'}(t)$. For $h \neq 1$, this follows from the expression (26) applied to the product of the polynomials $S_{I-3\rho(q)}(t)$ and $S_{2\rho(q)}(t)$ and multiplied by $V_q(t)$. For $h = 1$, this follows from the expressions (26) and (27).

Thus, $Z_I(t) = \Delta_I(t) + \sum_{I' \rhd I} \alpha_{I'} \Delta_{I'}(t) + \sum_{I'' \rhd I} \alpha_{I''} \Delta_{I''}(t)$, where the I' are nonsingular k-partitions and the I'' are singular ones. The chain $c(t) = Z_I(t) - \widehat{\varepsilon}_I(t) - \sum_{I' \rhd I} \alpha_{I'} \widehat{\varepsilon}_{I'}(t)$ is a stable cycle by Proposition 5.3.

The definition of $\widehat{\varepsilon}_I$ implies that $\widehat{\varepsilon}_I(t) = e_I(t) + \sum_{\widetilde{I}} \beta_{\widetilde{I}} e_{\widetilde{I}}$, where the k-partitions $\widetilde{I}$ are singular. Therefore, c is a linear combination of singular monomials. Since c is a stable cycle, it follows by Theorem 4.3 that $\langle c, \mathrm{E}_1(k) \rangle = 0$. On the other hand, $\langle c, e_I \rangle = 0$ for each nonsingular k-partition I. By Theorem 2.4, the set of such cycles e_I, together with a basis of $\mathrm{E}_1(k)$, forms a basis of the space $C_*(k)$. Hence $c = 0$. This completes the proof of formula (30). □

7. Laplace Operators of the Algebras $L^{(h)}(1)$

Definition 7.1. The endomorphism $\Gamma_k^{(h)} = d_k \delta_k + \delta_k d_k$ of the space $C_*(k; \mathbb{R}) = C_*(k) \otimes \mathbb{R}$ is called the *Laplace operator*, and the elements of its kernel are called the *harmonic chains* of the algebra $L(k)$.

Obviously, $\Gamma_k^{(h)}$ is a self-adjoint operator commuting with d_k and δ_k. The complex $C_*(k; \mathbb{R})$ is the direct sum of eigenspaces of $\Gamma_k^{(h)}$, which are subcomplexes of $C_*(k; \mathbb{R})$. For nonzero eigenvalues, these complexes are acyclic, and the space of harmonic chains is isomorphic to $H_*(k; \mathbb{R})$. This is a standard general claim (see [4]).

Definition 7.2 (B. L. Feigin)**.** A linear endomorphism A of the vector space $\Lambda(V)$ is called a *second-order operator of degree* m if $A(\Lambda^q(V)) \subset \Lambda^{q+m}(V)$ and

$$A(v_1 \wedge \cdots \wedge v_q) = \sum_{1 \leqslant i < j \leqslant q} (-1)^{i+j-1} A_{i,j}(v_1 \wedge \cdots \wedge v_q) - (q-2) \sum_{1 \leqslant i \leqslant q} A_i(v_1 \wedge \cdots \wedge v_q)$$

for arbitrary $v_1, \dots, v_q \in V$ ($q \geqslant 2$), where $A_{i,j}(v_1 \wedge \cdots \wedge v_q) = A(v_i \wedge v_j) \wedge v_1 \wedge \cdots \wedge \widehat{v}_i \wedge \cdots \wedge \widehat{v}_j \wedge \cdots \wedge v_q$ and $A_i(v_1 \wedge \cdots \wedge v_q) = v_1 \wedge \cdots \wedge A(v_i) \wedge \cdots \wedge v_q$.

One can readily prove that

$$A(u_1 \wedge \cdots \wedge u_q) = \sum_{1 \leqslant i < j \leqslant q} (-1)^{\omega(i,j)} A_{i,j}(u_1 \wedge \cdots \wedge u_q) - (q-2) \sum_{1 \leqslant i \leqslant q} A_i(u_1 \wedge \cdots \wedge u_q) \qquad (31)$$

for $u_1 \in \Lambda^{r_1}(V), \dots, u_q \in \Lambda^{r_q}(V)$ and $q \geqslant 2$, where $\omega(i,j) = r_i(r_1 + \cdots + r_{i-1}) + r_j(r_1 + \cdots + r_{j-1}) - r_i r_j$.

Lemma 7.3. $\Gamma_k^{(h)}$ *is a second-order operator of degree* 0.

Proof. The proof goes by a formal verification that uses only the fact that d_k is a second-order operator and δ_k acts by formula (8). □

Now let us study the Laplace operator $\Gamma^{(h)} = \Gamma_1^{(h)}$ of the algebra $L^{(h)}(1)$. For $I = [i_1, \dots, i_q]$, let $\kappa_h(I) = \kappa_h(i_1) + \cdots + \kappa_h(i_q)$ and $(I)_h = (i_1)_h + \cdots + (i_q)_h$, where $\kappa_h(i)$ is defined by formula (2).

Theorem 7.4. *Let* $\lambda^{(h)}(I) = \kappa_h(I) - ((I)_h^2 - (I)_h)/2$. *Then*

$$\Gamma^{(h)}(\varepsilon_{[I;J]}) \approx \lambda^{(h)}(I)\, \varepsilon_{[I;J]}. \qquad (32)$$

Moreover, $\Gamma^{(h \neq 1)}(\varepsilon_{[I;J]}) = \lambda^{(h)}(I)\, \varepsilon_{[I;J]}$. *In particular*, *the spectrum of* $\Gamma^{(h)}$ *coincides with the set of numbers* $\lambda^{(h)}(I)$, *where* $I \in \mathrm{N}(1)$.

Proof. One can readily verify that $\Gamma^{(h)}(e_i) = \lambda^{(h)}(i)e_i$. Therefore, $\Gamma^{(h)}(\delta_1 e_i) = \lambda^{(h)}(i)\delta_1 e_i$. This proves formula (32) for $\varepsilon(1)$-monomials of length 1.

Let $D^{(h)}(e_I) = \Gamma^{(h)}(e_I) - \lambda^{(h)}(I)\, e_I$. A straightforward computation shows that

$$\langle D^{(h)}(e_i \wedge e_j), e_u \wedge e_v\rangle = \begin{cases} (3i)_h(v-u)_h & \text{if } i \leqslant u < v \leqslant j, \\ (3u)_h(j-i)_h & \text{if } u < i < j < v, \end{cases} \qquad i+j = u+v.$$

It follows that $\Gamma^{(h\neq 1)}(e_i \wedge e_j) = \lambda^{(h)}(i,j)\, e_i \wedge e_j$, because $(3m)_{h\neq 1} = 0$. Since $\Gamma^{(h)}$ is a second-order operator, we obtain $\Gamma^{(h\neq 1)}(e_I) = \lambda^{(h)}(I)\, e_I$. Thus, $\Gamma^{(h\neq 1)} = p - (e_0^2 - e_0)/2$, where the linear mappings e_0 and p are defined by the formulas $e_0(e_I) = (I)_h e_I$ and $p(e_I) = \kappa_h(I)e_I$. Keeping in mind that e_0 commutes with δ_1 and that p commutes with δ_1 for $h \neq 1$, we obtain formula (32).

Let $h = 1$. The directly verifiable formulas

$$\Gamma^{(1)}(e_i \wedge e_j) = \lambda^{(1)}(i,j)\, e_i \wedge e_j - 3(i+j) \sum_{1\leqslant r<i} r e_{i-r} \wedge e_{j+r} + 3i\, \delta_1 e_{i+j},$$

$$\Gamma^{(1)}(\delta_1 e_i \wedge e_j) = \lambda^{(1)}(i,j)\, \delta_1 e_i \wedge e_j - 3 \sum_{1\leqslant r<i} r\big((i+2j+2r)\, \delta_1 e_{i-r} \wedge e_{j+r} + (i-2r)\, e_{i-r} \wedge \delta_1 e_{j+r}\big)$$

prove (32) for the $\varepsilon(1)$-monomials $e_i \wedge e_j$ and $\delta_1 e_i \wedge e_j$, since $j - i \geqslant 3$. They also imply the desired result for the $\varepsilon(1)$-monomials $e_i \wedge \delta_1 e_j$ and $\delta_1 e_i \wedge \delta_1 e_j$, since

$$\Gamma^{(1)}(e_i \wedge \delta_1 e_j) = \Gamma^{(1)}(\delta_1 e_i \wedge e_j) - \delta_1 \Gamma^{(1)}(e_i \wedge e_j), \quad \Gamma^{(1)}(\delta_1 e_i \wedge \delta_1 e_j) = \delta_1 \Gamma^{(1)}(\delta_1 e_i \wedge e_j).$$

Thus, formula (32) is true for the $\varepsilon(1)$-monomials of length 2 for $h = 1$.

Let $\varepsilon_{[I;J]} = e_{(i_1)} \wedge \cdots \wedge e_{(i_q)}$ be an arbitrary $\varepsilon(1)$-monomial, where $q > 2$. In formula (31), let $A = \Gamma^{(h)}$ and $u_a = e_{(i_a)}$. Using the expressions obtained for the $\varepsilon(1)$-monomials of lengths 1 and 2 and Lemma 2.3, we see that $\Gamma^{(1)}(\varepsilon_{[I;J]}) \approx \lambda(I)\, \varepsilon_{[I;J]}$, where

$$\lambda(I) = \sum_{1\leqslant a<b\leqslant q} \lambda^{(1)}(i_a, i_b) - (q-2) \sum_{1\leqslant a\leqslant q} \lambda^{(1)}(i_a) = \lambda^{(1)}(I).$$

Formula (32) now follows by Theorem 2.4. □

Corollary 7.5. $\dim H_q(L^{(h)}(1)) = 2$ *for* $q > 0$.

Proof. In view of the one-to-one correspondence between homology classes and harmonic chains, it suffices to show that $\lambda^{(h)}(I) = 0$ for $I \in \mathbf{N}(1)$ only if $I = \xi(1,q)$ or $I = \xi(2,q)$.

For $h = 1$, the claim follows from the easy-to-verify formula

$$\lambda^{(1)}(I) = \frac{1}{6}\bigg(\|I\|(i_1-1)(i_1-2) + \sum_{1\leqslant a<q} \|I\|_{a+1}(i_a + i_{a+1})(i_{a+1} - i_a - 3)\bigg), \tag{33}$$

where $\|I\|_m = i_m + \cdots + i_q$. Let $h \neq 1$. Since $\lfloor \frac{i+1}{3} \rfloor = \frac{i-(i)_h}{3}$, it follows from Theorem 7.4 that $\lambda^{(h\neq 1)}(I) = 0$ only if

$$\|I\| = (3(I)_h^2 - (I)_h)/2.$$

Since $\|I\| \geqslant (3q^2 - q)/2$ and $|(I)_h| \leqslant q$, we obtain $(I)_h = \pm q$. Thus, $I = \xi(1,q)$ or $I = \xi(2,q)$. □

Corollary 7.6. *The equation* $\lambda^{(1)}(I) = \lambda > 0$*, where* $I \in \mathbf{N}(1)$*, has at most finitely many solutions. Let* $I_1, \ldots, I_{r(\lambda)}$ *be all of its solutions, and let* $m(\lambda)$ *be the multiplicity of the eigenvalue* λ *of* $\Gamma^{(1)}$*. Then* $m(\lambda) = 2^{\operatorname{ind}_1 I_1} + \cdots + 2^{\operatorname{ind}_1 I_{r(\lambda)}}$.

Proof. Since $I \in \mathbf{N}(1)$, we see that the vector

$$v = ((i_1-1)(i_1-2), i_2 - i_1 - 3, \ldots, i_q - i_{q-1} - 3) = (v_1, \ldots, v_q)$$

is nonzero, and its coordinates are nonnegative by (33). It follows from the same formula that $\lambda = \lambda^{(1)}(I) > \frac{1}{6}\sum_{a=1}^{q} \|I\|_a v_a$. For $\lambda > 0$, this inequality has at most finitely many solutions on the set of triples $\{q, I, v\}$, where I is a partition, $v \neq 0$ is an integer vector with nonnegative coordinates, and $|I| = \dim(v) = q$. This proves the first claim.

If $\lambda^{(1)}(I)=\lambda$ and V_I is the space spanned by the vectors $\varepsilon_{[I;J]}$, then

$$m(\lambda)=\dim \bigoplus_{I\in\mathbf{N}(1),\,\lambda^{(1)}(I)=\lambda>0} V_I.$$

Since this sum is finite and $\dim(V_I)=2^{\mathrm{ind}_1 I}$, we obtain the desired formula for $m(\lambda)$. □

Corollary 7.7. *One has $\Gamma^{(h)}(\mathrm{St}_m(1))\subset \mathrm{St}_m(1)$. Moreover, $\Gamma^{(h\neq 1)}(\widehat{\varepsilon}_{[I;J]})=\lambda^{(h)}(I)\,\widehat{\varepsilon}_{[I;J]}$ and*

$$\Gamma^{(1)}(\widehat{\varepsilon}_{[I;J]})=\lambda^{(1)}(I)\,\widehat{\varepsilon}_{[I;J]}+\sum_{[I';J']\,\mathrm{tr}[I;J]}\alpha_{[I';J']}\,\widehat{\varepsilon}_{[I';J']}. \tag{34}$$

In particular, the multiplicity of an eigenvalue λ of $\Gamma^{(1)}$ on the space of stable cycles of L_1 is equal to the number of solutions of the equation $\lambda^{(1)}(I)=\lambda$, where $I\in\mathbf{N}(1)$.

Proof. Since $\Gamma^{(h)}(\mathrm{E}_m(1))\subset \mathrm{E}_m(1)$ by Theorem 7.4, we obtain the first claim, because

$$\langle\Gamma^{(h)}(\mathrm{St}_m(1)),\mathrm{E}_{m+1}(1)\rangle=\langle\mathrm{St}_m(1),\Gamma^{(h)}(\mathrm{E}_{m+1}(1))\rangle=0.$$

For $h\neq 1$, the second claim is obvious, because $\Gamma^{(h\neq 1)}$ acts diagonally in the basis of $\varepsilon(1)$-monomials.

For $h=1$ and $\widehat{\varepsilon}_{[I;J]}\in\mathrm{St}_m(L_1)$, the first claim implies the formula

$$\Gamma^{(1)}(\widehat{\varepsilon}_{[I;J]})=\sum_{[I';J'],\,\|I'\|=\|I\|,\,|J'|\geqslant m}\alpha_{[I';J']}\,\widehat{\varepsilon}_{[I';J']},$$

where $\alpha_{[I';J']}=\langle\Gamma^{(1)}(\widehat{\varepsilon}_{[I;J]}),\varepsilon_{[I';J']}\rangle=\langle\widehat{\varepsilon}_{[I;J]},\Gamma^{(1)}(\varepsilon_{[I';J']})\rangle$.

Since $\Gamma^{(1)}(\varepsilon_{[I';J']})=\lambda^{(1)}(I')\varepsilon_{[I';J']}+\dots$ by formula (32), we see that $\alpha_{[I;J]}=\lambda^{(1)}(I)$ and $\alpha_{[I';J']}=0$ if either $[I';J']\lhd[I;J]$ or $[I;J]$ and $[I';J']$ are incomparable. □

Corollary 7.8 (cf. [1])**.** *The stable cycle $Z_I(t)=S_{I-3\rho(q)}(t)V_q^3(t)\in C_*(L_1)$, where $I\in\mathrm{N}(1)$ is a maximal partition with respect to $\trianglelefteq$, is an eigenvector of $\Gamma^{(1)}$. In particular, the harmonic chains of dimension $q>0$ of L_1 are exhausted by the chains $Z_{\xi(1,q)}(t)$ and $Z_{\xi(2,q)}(t)$.*

Proof. The maximality of I and formulas (30) imply that $Z_I(t)=\widehat{\varepsilon}_I$. By the same cause, formula (34) shows that $\widehat{\varepsilon}_I$ is an eigenvector of $\Gamma^{(1)}$. □

Remark 7.9. For the algebras $L^{(h)}_{-1,0}$, the harmonic chains are exhausted by the chains $e_{-1}\wedge e_0\wedge e_1$ and e_0, respectively. One can show that for $k\geqslant 1$

$$\Gamma_k^{(h\neq 1)}=p+\frac{1}{2}\Bigg((k+1)_h^2\,e_0-e_0^2-\sum_{r=1}^{k-1}(e_re_r^*+e_r^*e_r)\Bigg),\quad \text{where }\ p(e_I)=\frac{\|I\|-(I)_h}{3}\,e_I$$

and e_r^* is the operator dual to the adjoint action of e_r. This implies that

$$\Gamma_k^{(h\neq 1)}=\begin{cases} p+\frac{1}{2}(e_{-1}e_1+e_0^2+e_1e_{-1}) & \text{if } k=-1,\\ p+\frac{1}{2}(e_0^2+e_0) & \text{if } k=0,\\ p-\frac{1}{2}(e_0^2-e_0) & \text{if } k=1,\\ p-\frac{1}{2}(e_{-1}e_1+e_0^2+e_1e_{-1}) & \text{if } k=2.\end{cases}$$

It is not difficult to obtain the spectral resolutions of $\Gamma_k^{(h\neq 1)}$ from these formulas for $k=-1,0,1,2$. One can show that the chains $e_1^r(e_2\wedge e_5\wedge\cdots\wedge e_{3q-1})$, where $r=0,1,\dots,2q$, is a basis of the space of q-dimensional harmonic chains of $L_2^{(h\neq 1)}$ and that the stable filtration is invariant under the $\Gamma_2^{(h\neq 1)}$-action.

The spectra of $\Gamma_k^{(1)}$ for $k\neq 0,1$ and of $\Gamma_k^{(h\neq 1)}$ for $k>2$ contain complicated irrational values. But hopefully there exists a tame set of polynomials parameterized by h and k whose set of roots coincides with these spectra. The multiplicities of the eigenvalues of the operators $\Gamma_k^{(1)}$ are likely to be finite for all $k\geqslant -1$.

8. Addendum: A Combinatorial Proof of Theorem 3.6

Let us construct a bijective map $S\colon \mathrm{D}_q^{(n)} \to \mathbf{N}_q^{(n)}$. Recall that the *diagram of a partition* $I=[i_1,\dots,i_q]$ is the set of points $(a,b)\in\mathbb{Z}\times\mathbb{Z}$ (*vertices*) such that $1\leqslant b\leqslant q$ and $1\leqslant a\leqslant i_b$. The *diagonal of the partition* I is the set of vertices with $a+b=q+1$.

Assume that the diagonal of I contains r vertices. Let us number them from bottom to top. Let x_i be the number of diagram vertices located in the row to the right of the ith diagonal vertex, including this vertex, and let y_i be the number of diagram vertices located in the column strictly below the ith vertex. We can interpret I as a pair of integer sequences

$$I=(x_1,\dots,x_r\mid y_1,\dots,y_r),\quad \text{where } 1\leqslant x_1<\dots<x_r,\ 0\leqslant y_1<\dots<y_r.$$

Such a pair corresponds to a strict partition if and only if

(1) $x_{i+1}-x_i\geqslant 2$ for $i=1,\dots,r-1$.

(2) $y_{i+1}-y_i=1$ or 2 for $i=1,\dots,r-1$.

(3) $y_1=0$ or 1.

(4) If $x_1=1$, then $y_1=0$.

Let $I=(x_1,\dots,x_r\mid y_1,\dots,y_r)\in\mathrm{D}_q^{(n)}$, and let $1\leqslant a_1<\dots<a_s\leqslant r$ be all the numbers such that $y_{a_k}-y_{a_k-1}=2$. (By definition, $y_0=-1$.) Set $S(I)=[\widetilde{I};\widetilde{J}]$, where

$$\widetilde{I}=[x_1+y_1,\dots,x_r+y_r],\qquad \widetilde{J}=[x_{a_1}+y_{a_1},\dots,x_{a_s}+y_{a_s}].$$

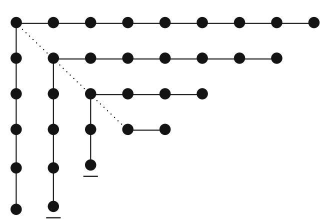
Since $s=q-r$, it follows from conditions (1)–(4) that $S(I)\in\mathscr{N}_q^{(n)}$.

Now let us define a mapping $S^{-1}\colon\mathbf{N}_q^{(n)}\to\mathrm{D}_q^{(n)}$. Let $[I;J]\in\mathbf{N}_q^{(n)}$, where $I=[i_1,\dots,i_{|I|}]$. Set $S^{-1}[I;J]=(x_1,\dots,x_{|I|}\mid y_1,\dots,y_{|I|})$, where

$$y_1=\begin{cases}0 & \text{if } i_1\notin J,\\ 1 & \text{if } i_1\in J,\end{cases}\qquad y_a=\begin{cases}y_{a-1}+1 & \text{if } i_a\notin J,\\ y_{a-1}+2 & \text{if } i_a\in J,\end{cases}\qquad 2\leqslant a\leqslant|I|,$$

and $x_a=i_a-y_a$. Obviously, $|S^{-1}[I;J]|=y_{|I|}+1=|I|+|J|=q$. Therefore, $S^{-1}[I;J]\in\mathrm{D}_q^{(n)}$. The definitions imply that the mappings S and S^{-1} are mutually inverse. The proof of Theorem 3.6 is complete.

Example: for the partition $I=[2,3,5,6,8,9]=(2,4,7,9\mid 0,2,4,5)$ with diagram



the parts of $S(I)$ are the numbers of vertices connected by the solid lines. Such a part is marked if the lowest corresponding vertex of the diagram is located strictly below the diagonal and there are no more vertices in the row to the right of this vertex. Thus, $S[2,3,5,6,8,9]=[2,\underline{6},\underline{11},14]$.

Remark 8.1. Using Lemma 3.10 and the obtained combinatorial interpretation of Sylvester's identity, one can prove the following generalization:

$$\prod_{i=k}^{\infty}(1+tx^i)=1+tx^k\,\frac{1+tx^{k+1}}{1-x}+\sum_{l=2}^{\infty}t^lx^{kl+3l(l-1)/2}\,\frac{(1+tx^k)\cdots(1+tx^{l+k-2})(1+tx^{2l+k-1})}{(1-x)\cdots(1-x^{l-1})(1-x^l)}\,.$$

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Universität Bern, Institut für Anatomie
e-mail: Weinstein@ana.unibe.ch

Translated by F. V. Weinstein